The Sunada Method

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In his famous paper [**Su**], Sunada gave a method to construct pairs of isospectral manifolds. His method was based on an interpretation of isospectrality in terms of finite groups.

The simplicity and elegance of the Sunada method led to a period of many significant developments in the understanding of isospectral manifolds and related problems. It would thus seem appropriate to mark the fifteenth anniversary of the appearance of [**Su**] by taking stock of the "state of the art" of the Sunada method today.

This is the goal of the present paper. We would particularly like to focus on some recent applications of the Sunada method, and hope to point out some possible directions for further growth.

In §1, we place the Sunada method in a historical context. In §2, we present a number of different proofs of the Sunada Theorem. In §3, we discuss some recent applications of the Sunada method to scattering theory. In §4, we discuss the Sunada method in the context of graph theory.

In the course of §3 and §4, we will see that each of the proofs that are given in §2 plays an important and distinctive role. The fact that the Sunada

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*Partially supported by grants from the Israel Academy of Sciences, the Fund for the Promotion of Research at the Technion, and the New York Metropolitan Research Fund construction has so many different interpretations, as expressed in these quite different proofs, allows it to play an important role in a number of quite different areas.

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1 A Historical Discussion

We begin with the following definition:

Definition 1.1 *Let G be a finite group, and H_1, H_2 two subgroups of G. Then the triple (G, H_1, H_2) satisfies the Sunada condition if*

$$(\dagger) \quad \textit{for all } g \in G, \#\{[g] \cap H_1\} = \#\{[g] \cap H_2\},$$

where "$[g]$" denotes the conjugacy class of g in G.

It is not difficult to see that:

Lemma 1.1 *The Sunada condition $(\dagger)$ is equivalent to the following condition:*

$$(\ddagger) \quad \mathit{ind}_{H_1}^G(\mathbf{1}_{H_1}) \quad \textit{is } G\textit{-equivalent to } \mathit{ind}_{H_2}^G(\mathbf{1}_{H_2}),$$

where "$\mathbf{1}_{H_i}$" denotes the trivial representation of the group H_i, and "ind" denotes the induced representation.

Proof: We will use the fact that two G-representations are G-equivalent if and only if they have the same characters. But, for $g \in G$,

$$\operatorname{tr}\left[g : \operatorname{ind}_{H_1}^G(\mathbf{1}_{H_1}) \to \operatorname{ind}_{H_1}^G(\mathbf{1}_{H_1})\right] = \frac{\#\{g' : gg' = g'h \text{ for some } h \in H_1\}}{\#(H_1)},$$

and similarly for H_2, as can be seen by choosing the obvious basis for $\operatorname{ind}_{H_1}^G(\mathbf{1}_{H_1})$.

The lemma will then follow once we show that the condition $(\dagger)$ is equivalent to:

$$\#\{g' : gg' = g'h \text{ for some } h \in H_1\} = \#\{g' : gg' = g'h \text{ for some } h \in H_2\}.$$

But

$$\#\{g' : (g')^{-1}gg' \in H_1\} = \#([g] \cap H_1) \cdot \#(C_G(g)),$$

where $C_g(g)$ denotes the centralizer of g in G. The conclusion follows.
■

The condition (†) seems to have been first studied in the context of algebraic number theory. The following result appears as an exercise in **[CF]**, as was pointed out in **[Su]**:

Theorem 1.1 *Let L_0 be a number field, K a Galois extension of L_0, and L_1 and L_2 two subfields of K containing L_0.*

Suppose $G = Gal(K/L_0)$ and $H_i = Gal(K/H_i)$, $i = 1, 2$, satisfy (†). *Then the zeta functions $\zeta_{L_1/L_0}(s)$ and $\zeta_{L_2/L_0}(s)$ are equal.*

It was shown by Perlis in **[Per]** that the sufficiency condition in Theorem 1.1 is also a necessary condition. Hence the condition (†) gives a complete and satisfying answer to the question of when two number fields share the same zeta function, except for the very difficult problem of determining when a given group arises as the Galois group of a Galois extension of a given field.

It was pointed out to us recently by Tolya Vershik that the condition (‡) appears in the paper **[Ma]** of George Mackey. Mackey was motivated by the question of when one can deduce the "classical mechanics" of a group action from its "quantum mechanics," in the following sense: given a group G acting on measure spaces X_1 and X_2 by measure-preserving transformations, suppose that $L^2(X_1)$ is G-equivalent to $L^2(X_2)$. Does it follow that there is a measurable G-equivalence $\phi : X_1 \to X_2$? He observes that any triple of finite groups satisfying (‡) where H_1 is not conjugate to H_2 in G will provide a counterexample, since a G-equivalence $G/H_1 \to G/H_2$ is precisely a conjugation of H_1 to H_2 in G.

The condition (‡) was quite actively studied by researchers in the classification of finite groups, for obvious reasons. Our favorite reference for this is the paper **[Gu]**. See **[CIM]** for a discussion of various triples of groups satisfying (†).

In **[Su]**, Sunada proved the following

Theorem 1.2 ([Su]) *Let (G, H_1, H_2) satisfy* (†), *and let $\phi : \pi_1(M) \to G$ be a surjective homomorphism.*

If M^{H_1} and M^{H_2} are the coverings of M with

$$\pi_1(M^{H_i}) = \phi^{-1}(H_i),$$

then M^{H_1} is isospectral to M^{H_2}.

We will present four proofs of Sunada's Theorem in the next section.

At about the same time, Gordon and Wilson [**GW**] constructed the first examples of isospectral deformations of metrics. Their examples were 2-step nilmanifolds. The connection between their work and Sunada's was not made clear until somewhat later, in the papers [**DG**], where they gave a version of the Sunada Theorem for Lie groups. 2-step nilpotence enters the picture because the condtion (†) may be realized by triples $(G, H, \phi(H))$, where ϕ satisfies the condition that $\phi(g)$ is conjugate to g for all g. Such automorphisms are called almost-inner automorphisms. It is not difficult to construct almost-inner automorphisms of 2-step nilpotent groups which are not inner, essentially by linear algebra. In this way, one sees that 2-step nilpotent Lie groups provide a rich source of almost-inner automorphisms which are not inner.

As we mentioned in [**SGCC**], the graph theorists seem never to have discovered the Sunada method. One reason for this is that they had already discovered a number of different isospectral constructions. The first to observe that the Sunada Theorem applies to graphs was Peter Buser [**Bu1**].

The application of the Sunada technique to construct isospectral graphs with different geometric properties is still somewhat underdeveloped. We will present some new examples in §4 below.

2 Four Proofs of the Sunada Theorem

Sunada's original proof of his theorem was a simple but elegant application of the trace formula for finite groups. A slight variant of it is given in [**CIM**], and goes as follows: Let M^G denote the covering of M with $\pi_1(M^G) = \phi^{-1}(1)$. If H_t^N denotes the heat kernel on the manifold N, then classically one has that

$$H_t^{M^{H_i}i}(x, y) = \sum_{h \in H_i} H_t^{M^G}(\widetilde{x}, \widetilde{y}),$$

where $\widetilde{x}$ denotes a point in M^G lying over x.

We thus have

$$\operatorname{tr}(H_t^{M^{H_i}}) = \frac{1}{\#(H_i)} \sum_{h \in H_i} \int_{M^G} H_t(\widetilde{x}, h(\widetilde{x}))\; d\widetilde{x}.$$

Noting that, in the sum on the right, G-conjugacy classes give equal terms, we have

$$\operatorname{tr}(H_t^{M^{H_i}}) = \frac{1}{\#(H_i)} \sum_{[g]} \#([g] \cap H_i) \int_{M^G} H_t^{M^G}(\widetilde{x}, h(\widetilde{x}))\; d\widetilde{x}.$$

The fact that the right-hand side is independent of i now establishes the theorem.

■

Sunada also observed the following argument, which is a direct translation of the argument from algebraic number theory, and is valid in situations where one knows that length isospectrality implies isospectrality (for instance, graphs or manifolds of negative curvature):

Let γ_1 be a closed geodesic on M^{H_1}, and let γ_0 denote its image in M. If γ_0 is primitive (that is, not the same path traversed several times), the number of times γ_0 lifts to a closed geodesic in M^{H_i} is

$$\frac{\#(g : g^{-1} g_0 g \in H_1)}{\#(H_1)} = \operatorname{tr}\left(g_0 : L^2(G/H_1) \to L^2(G/H_1)\right),$$

where g_0 is a representative of $[\gamma_0]$ in G. Thus, the equivalence of (†) with (‡) gives a bijection between closed geodesics lying over γ_0 in M^{H_1} and M^{H_2}.

When γ_0 is not primitive, one must modify this argument slightly, but the modifications only depend on G, and not on H_i, $i = 1, 2$.

■

A third argument, known as transplantation, was observed by Buser in **[Bu1]**, and developed by Bérard in **[Be]**. It begins with the observation that we may consider M to be obtained from a fundamental domain F by gluing the faces of copies of F in pairs by isometries g_j which generate $pi_1(M)$. In this way, M^{H_i} may be obtained from the Schreier graph of G/H_i with respect to the generators $\phi(g_j)$, by gluing one copy of F to each coset of G/H_i, and gluing the faces according to the action of g_j.

We may thus think of a function on M^{H_i} as a function on G/H_i with values in the functions on F.

The condition (‡) then gives a G-equivariant isomorphism

$$L^2(G/H_1) \to L^2(G/H_2),$$

and hence a map

$$L^2(M^{H_1}) \to L^2(M^{H_2})$$

which commutes with the process of crossing from one copy of the fundamental domain to an adjacent one.

A moment's reflection shows that the G-invariance implies that this map takes smooth functions to smooth functions, and commutes with the Laplacian. Indeed, on each point interior to the fundamental domain, the map takes a function to linear combinations of the function restricted to different copies of the fundamental domain, hence preserves smoothness. On the other hand, G-equivariance tells us that the map is independent of the choice of fundamental domain, so the map is smooth at all points.

Isospectrality follows, since we have shown explicitly how to take eigenfunctions on M^{H_1} and cut them up and reassemble them on M^{H_2}.

■

A fourth argument, due to Hubert Pesce **[Pe3]**, runs as follows: Let λ denote an eigenvalue of M^G, and E_λ the associated eigenspace. Then E_λ is a representation space of G, and the multiplicity of λ in $\mathrm{spec}(M^{H_i})$ is just the dimension of the H_i-invariant subspace of E_λ.

Writing this last as $[\mathbf{1}_{H_i} : \mathrm{Res}^G_{H_i}]$, where Res denotes the restriction of the representation and $[V : W]$ denotes the multiplicity of the representation V in W, Frobenius reciprocity says that

$$[\mathbf{1}_{H_i} : \mathrm{Res}^G_{H_i}(E_\lambda)] = [\mathrm{ind}^G_{H_i}(1_{H_i}) : E_\lambda].$$

This last is independent of i, since, by the equivalence of (†) with (‡), we have that $\mathrm{ind}^G_{H_1}(\mathbf{1}_{H_1})$ is G-equivalent to $\mathrm{ind}^G_{H_2}(\mathbf{1}_{H_2})$.

■

It is worth remarking that this proof shows something stronger than the Sunada Theorem. Namely, if one replaces the condition (‡) with the condition:

$$[\mathrm{ind}_{H_1}^{G}(\mathbf{1}_{H_1}) : E_\lambda] = [\mathrm{ind}_{H_2}^{G}(\mathbf{1}_{H_2}) : E_\lambda] \text{ for all } E_\lambda,$$

then one gets a necessary and sufficient condition for isospectrality of M^{H_1} and M^{H_2}. In effect, some representations of G need not arise in the spectrum of M^G, and hence the condition (‡) need not be verified at these representations.

3 Isoscattering

It was apparent early on that the Sunada method applied to many more operators than just the Laplacian. Indeed, one sees from the first proof of the Sunada Theorem that the method applies to any self-adjoint operator L which has the following properties:

(i) L is natural with respect to finite coverings.

(ii) L is of trace class.

Using this idea, we were able to prove in **[IP]** that:

Theorem 3.1 *There exists a Riemann surface S and two potentials ϕ_1 and ϕ_2, such that the Schrodinger operators $\Delta + \phi_1$ and $\Delta + \phi_2$ are isospectral.*

The idea of the proof was to place a Sunada triple (G, H_1, H_2) inside a larger group G' such that H_1 and H_2 are conjugate inside G' but not inside G. Lifting the metric structure from G' but the potential only from G produces two operators on two surfaces such that the surfaces are isometric, but the isometry does not respect the operators.

We remark that the following problem has stubbornly insisted on remaining open for some time:

Problem 3.1 *Find a Riemann surface (constant curvature -1) such that, for all potentials ϕ, the spectrum of $\Delta + \phi$ determines ϕ uniquely.*

It was shown by Guillemin and Kazhdan **[GK]** that this property holds for a generic surface (not constant curvature), but their proof fails for *all* constant curvature surfaces. One expects that a generic Riemann surface has this property.

In recent joint work with Ruth Gornet and Peter Perry [**BGP**], we have extended the Sunada method to study the poles of the scattering operator on geometrically finite hyperbolic manifolds. We will not enter into a discussion of the technicalities of scattering theory, see [**BGP**] for details. We do observe, however, that the poles of the scattering operator play a role analogous to the eigenvalues of the Laplacian for non-compact hyperbolic manifolds. So isoscattering manifolds may be regarded as non-compact analogues of isospectral manifolds.

The Sunada method had previously been used to construct examples of isoscattering in the setting of Riemann surfaces of finite and infinite area, see [**Ber**], [**GZ**], and [**Z**].A crucial obstacle to extending the Sunada method to the case of the scattering operator for hyperbolic manifolds of dimension $ge3$ is that the scattering operator is not of trace class. It is therefore of fundamental importance in these cases that the transplantation argument (proof # 3 in §2) is available. We remark that the papers [**Ber**] and [**Z**] already made use of transplantation in their setting, as the technical simplicity and clarity of transplantation was already apparent in these cases.

The main result of [**BGP**] is then:

Theorem 3.2 *There exist two hyperbolic* 3*-manifolds* M_1 *and* M_2 *such that:*

(i) M_1 *and* M_2 *are isoscattering.*

(ii) The boundaries at infinity of M_1 *and* M_2 *are conformally equivalent.*

(iii) M_1 *and* M_2 *are not isometric.*

The idea beyond transplantation that one needs to establish the theorem is to adapt the construction of Theorem 3.1. One constructs M_1 and M_2 as Schottky manifolds. Lifting the conformal structure at infinity from G' while lifting the Schottky structure from G gives the desired manifolds M_1 and M_2.

It is evident that many of the constructions of isospectral Riemann surfaces go through to the category of isoscattering manifolds, even if one has to be occasionally clever in the translation. Here are some problems that strike us as interesting:

Problem 3.2

(i) The original examples of Vignéras **[V]** *are Riemann surfaces which are isospectral, but which are not related by the Sunada method. They are, however, trace equivalent (see the next section for a discussion).*

Are there examples of non-trivial hyperbolic manifolds which are isoscattering but whose boundaries are not related by the Sunada method?

(ii) Let M_1 and M_2 be two geometrically finite hyperbolic 3-manifolds which are isoscattering. Their boundaries ∂M_1 and ∂M_2 carry a natural conformal structure. Place on ∂M_1 and ∂M_2 the metrics of constant curvature -1 in their conformal class. Must ∂M_1 and ∂M_2 be isospectral in these metrics?

This of course will be the case if M_1 and M_2 are related by the Sunada method. So this question may be regarded as a refinement of (i).

4 Isospectral Graphs

A natural question is to ask to what extent the Sunada method accounts for isospectrality in general.

This question has been examined from many perspectives by Hubert Pesce [**1** − **3**]. We heartily recommend the survey [**Pe2**] for a comprehensive account. We will not go over this material again, except to emphasize the following points: the initial examples of isospectral manifolds due to Vignéras [**V**] and Ikeda [**Ik**] do not arise from the Sunada construction directly. However, the Vignéras examples may be understood by broadening the second proof of the Sunada Theorem to include representationally equivalent manifolds. Pesce then showed that all isospectral hyperbolic manifolds were representationally equivalent [**Pe2**]. Furthermore, the Ikeda examples could be understood as arising from the fourth proof of the Sunada method.

Quite recently, Gordon, Gornet, Schueth, Webb, and Wilson ([**GW2**] and [**GGSWW**]) found a new technique for constructing isospectral manifolds using Riemannian submersions, which seems to be a radical departure from the Sunada method. In particular, in [**Sc**] D. Schueth gives the first example of isospectral closed manifolds which are simply connected.

In the paper **[NS]**, we considered the analogous question for graphs. There we proved that every pair of isospectral graphs arises from a weakened form of the Sunada construction, but also exhibited pairs of graphs which do not arise from a strong version of the Sunada method. See **[NS]** for precise statements.

Here is a new construction of isospectral graphs which is quite different from the Sunada construction. It is a special case of a more general construction of isospectral graphs which need not be regular due to Fujii and Katsuda **[FK]**. The proof of the general case in **[FK]** involves a computation of the eigenvalues. In our more specific case, a vastly simpler path-counting argument is available.

With some imagination, this construction may be viewed as the graph-theoretic analogue of the Riemannian submersion construction.

Definition 4.1 *Let B be a k_1-regular graph, with vertices $x_1, \ldots, x_N$, and for each $i = 1, \ldots, N$, let G_i be an isospectral k_2-regular graph, with the size M of G_i, G'_i independent of i.*

The graph expansion *$B((G_i))$ of B by (G_i) is the graph given as follows:*

- *(i) The vertices of $B((G_i))$ are the pairs (x_i, y) with x_i a vertex in B and y a vertex in G_i.*
- *(ii) The vertices (x_i, y) and (x_j, z) are connected by an edge if only only if either (a) $x_i = x_j$ and y and z are connected by an edge in G_i, or (b) x_i and x_j are joined by an edge in B (and y and z arbitrary).*

It is clear that $B((G_i))$ is a regular graph of regularity $Nk_1 + k_2$. Furthermore, if N is large compared to k_2, it is an easy matter to reconstruct the graphs B and the G_i's from $B((G_i))$.

We then have:

Theorem 4.1 (compare **[FK]**) *Suppose that, for each i, the graph G_i is isospectral to G'_i. Then $B((G_i))$ is isospectral to $B((G'_i))$.*

Proof: We count closed paths of a given length. The number of closed paths of a given length which stay in a given G_i are the same as the number of closed paths of that length on G'_i, since the two graphs are isospectral. If a given path γ leaves one G_i, then we look at the collection of closed paths γ'

which have the property that if the j-th vertex of γ lies in G_i, then the j-th vertex of γ' lies in G_i (resp. G'_i). This set clearly has the same cardinality for $B((G_i))$ and $B((G'_i))$, because the condition that a path close up is essentially vacuous.

It follows that the two paths are length isospectral, and hence isospectral.
∎

As an immediate consequence, we have:

Corollary 4.1 *There exists k and, for each N a family of k-regular graphs $G_1^N, \ldots, G_{l_N}^N$, such that:*

(i) For each N, the graphs $G_1^N, \ldots, G_{l_N}^N$ are all k-regular, distinct, and mutually isospectral.

(ii) The size of the G_i^N's grows linearly with N.

(iii) The size of l_N grows exponentially in N.

Proof: We choose two isospectral k_1-regular graphs G_1 and G_2. For each N, we choose a k_2-regular graph B_N on $2N$ vertices which has no automorphisms. The graphs $B_N((H_i))$ where each H_i is either G_1 or G_2 are then isospectral by the theorem, and are all distinct because we can reconstruct B_N and the H_i's from the graph structure alone.
∎

A result analogous to Corollary 4.1 was established in [**BGG**]. It made use of a different isospectral construction known as Seidel switching. The graphs of [**BGG**] had loops and multiple edges, while those of Corollary 4.1 do not.

We close this section by showing how one may use the Sunada method to construct simple (without loops or multiple edges) regular isospectral graphs which have different geometric properties. We considered this problem in the case of the Cheeger constant in [**SGCC**]. We now consider the diameter.

Theorem 4.2 *There exists pairs of simple 3-regular graphs $\{G_i, G'_i\}$, such that G_i and G'_i are isospectral, $diam(G_i) \to \infty$ as $i \to \infty$, and*

$$\frac{diam(G_i)}{diam(G'_i)} \to 5/4 \text{ as } i \to \infty.$$

We begin with the following observation: let (G, H_1, H_2) be a Sunada triple, G_0 a simple regular graph, and $\phi : \pi_1(G_0) \to G$ any homomorphism. Then the covering graphs $G_1 = G_0^{H_1}$ and $G_2 = G_0^{H_2}$ are simple, regular graphs which are isospectral. Indeed, they are isospectral by the Sunada Theorem, and a covering of simple regular graphs is simple and regular.

We now consider the graphs shown in Figures 1 and 2 below, taken from **[Bu2]** and **[SGCC]**:

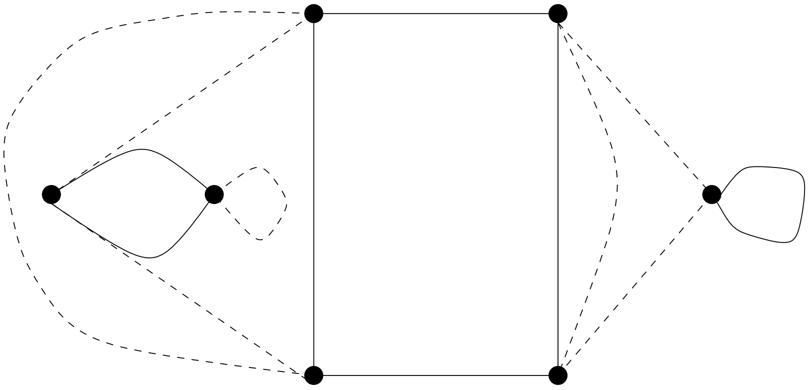

Figure 1: The first graph

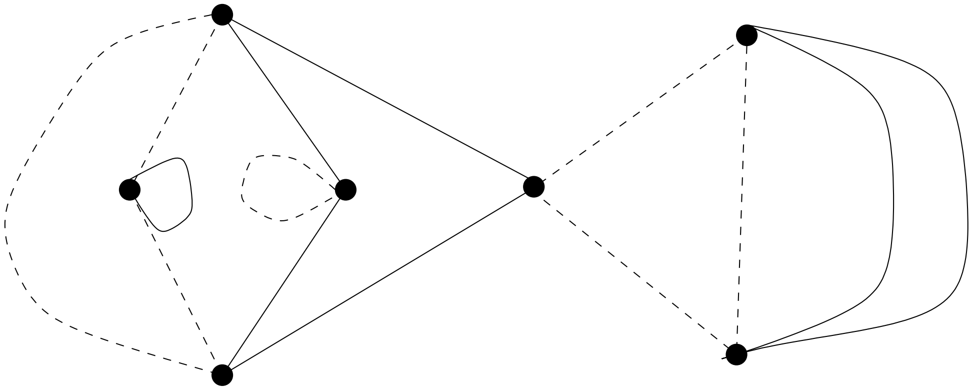

Figure 2: The second graph

These graphs are Cayley graphs of a Sunada triple, and hence isospectral.

One sees by inspection that the diameter of the first graph is 4, while the diameter of the second graph is 3. The different colorings of the lines denote different generators of the group.

We now consider, for each k, the 3-regular graph G_0^k shown in Figure 3 below.

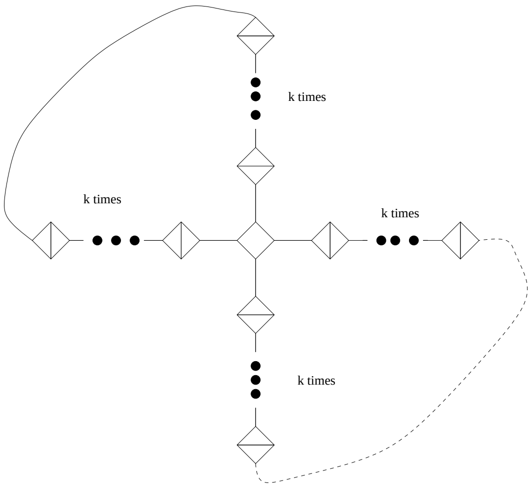

Figure 3: The graph G_0^k

This graph has the property that, if one deletes the two edges joining the corners, then any path joining the corner vertices must pass through the center, and hence be of length approximately (const)k.

We now apply the Sunada construction to the graph G_0^k, with the obvious homomorphism $\pi_1(G_0^k) \to G$. In practice, this means the following operation: we take seven copies of the graph G_0^k, and lay them out according to the vertices of the graph in Figure 1 (resp. Figure 2). We then break open the two edges joining the corners on each copy, and then join two half-edges of the same color together when the corresponding vertices in Figure 1 (resp. 2) are joined by an edge of that color.

The Sunada Theorem then guarantees that the pair of graphs constructed in this way for each k is isospectral. It is also a simple matter to compute the diameters of these graphs by inspection, in order to verify the theorem. In effect, one must traverse approximately (const)k edges in the corresponding graph for each vertex in a path of the graphs in Figures 1 and 2.

■

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